

# A Spectral Characterization of Strongly Distance-Regular Graphs with Diameter Four.

M.A. Fiol

<sup>b</sup>Universitat Politècnica de Catalunya, BarcelonaTech  
Dept. de Matemàtica Aplicada IV, Barcelona, Catalonia  
(e-mail: [fiol@ma4.upc.edu](mailto:fiol@ma4.upc.edu))

## Abstract

A graph  $G$  with  $d + 1$  distinct eigenvalues is called strongly distance-regular if  $G$  itself is distance-regular, and its distance- $d$  graph  $G_d$  is strongly-regular. In this note we provide a spectral characterization of those distance-regular graphs with diameter  $d = 4$  which are strongly distance-regular. As a byproduct, it is shown that all bipartite strongly distance-regular graphs with such a diameter are antipodal.

*Keywords:* Distance-regular graph; Strongly distance-regular graph; Spectrum.

*AMS subject classifications:* 05C50, 05E30.

## 1 Introduction

For background on distance-regular graphs and strongly regular graphs, we refer the reader to Brouwer, Cohen, and Neumaier [2], Brouwer and Haemers [3], and Cameron [4]. A *strongly distance-regular graph* is a distance-regular graph  $G$  (of diameter  $d$ , say) with the property that its distance- $d$  graph  $G_d$  is strongly regular. Known examples of strongly distance-regular graphs are the strongly regular graphs (since  $G_d$  is the complement of  $G$ ), the antipodal distance-regular graphs (where  $G_d$  is a disjoint union of complete graphs), and all the distance-regular graphs with  $d = 3$  and third largest eigenvalue  $\lambda_2 = -1$ . This last result was reported by Brouwer [1], and Brouwer, Cohen, and Neumaier [2] and, in fact, the same conclusion was reached by the author [7] by only requiring the regularity of  $G$ . In fact there are some infinite families of this type, such as the generalized hexagons and the Brouwer graphs (see, for instance, [2]).

The above situation suggests the open problem of deciding whether or not the above known families of strongly distance-regular graphs exhaust all the possibilities (see [8, Conjecture 3.6] or Cameron [5]). Going one step further in this direction, here we prove that a distance-regular graph  $G$  with five distinct eigenvalues  $\lambda_0 > \lambda_1 > \dots > \lambda_4$  (the

case of diameter four) is strongly distance regular if and only an equality involving them, and the intersection parameters  $a_1$  or  $b_1$ , is satisfied. Then, as a consequence, it is shown that all bipartite strongly distance-regular graphs with such a diameter are antipodal.

## 2 The result

In proving our result, we use the following scalar product:

$$\langle p, q \rangle_G = \frac{1}{n} \operatorname{tr}(p(\mathbf{A})q(\mathbf{A})) = \frac{1}{n} \sum_{i=0}^d m_i p(\lambda_i) q(\lambda_i), \quad p, q \in \mathbb{R}_d[x], \quad (1)$$

and the following lemma (see [6, 8]):

**Lemma 1.** *Let  $G$  be a distance-regular graph with spectrum  $\operatorname{sp} G = \{\lambda_0, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$ , where  $\lambda_0 > \lambda_1 > \dots > \lambda_d$ . Then,*

(a)  *$G$  is  $r$ -antipodal if and only if*

$$m_i = \frac{\pi_0}{\pi_i} \quad (i \text{ even}), \quad m_i = (r-1) \frac{\pi_0}{\pi_i} \quad (i \text{ odd}).$$

(b)  *$G$  is strongly distance-regular if and only if, for some positive constants  $\alpha, \beta$ ,*

$$m_i \pi_i = \alpha \quad (i \text{ odd}), \quad m_i \pi_i = \beta \quad (i \text{ even}, i \neq 0).$$

Now we are ready to prove our main result:

**Theorem 2.** *Let  $G$  be a distance-regular graph with  $n$  vertices, diameter  $d = 4$ , and distinct eigenvalues  $\lambda_0 (= k) > \lambda_1 > \dots > \lambda_4$ . Then  $G$  is strongly distance-regular if and only if*

$$(1 + \lambda_1)(1 + \lambda_3) = (1 + \lambda_2)(1 + \lambda_4) = -b_1. \quad (2)$$

Moreover, in this case,  $G$  is antipodal if and only if either,

$$\lambda_1 \lambda_3 = -k, \quad \text{or} \quad \lambda_1 + \lambda_3 = a_1 \quad (3)$$

*Proof.* Notice first that the multiplicities  $m_0 (= 1), m_1, \dots, m_4$ , satisfy the following equations:

$$\sum_{i=0}^4 m_i = n, \quad \sum_{i=0}^4 m_i \lambda_i = 0, \quad \sum_{i=0}^4 m_i \lambda_i^2 = nk, \quad \sum_{i=0}^4 m_i \lambda_i^3 = nka_1,$$

or, in terms of the scalar product (1),

$$\langle 1, 1 \rangle_G = 1, \quad \langle x, 1 \rangle_G = 0, \quad \langle x^2, 1 \rangle_G = k, \quad \langle x^3, 1 \rangle_G = ka_1. \quad (4)$$

From this, and expanding the product below, we have that

$$\langle (x - \lambda_0)(x - \lambda_2)(x - \lambda_4), 1 \rangle_G = ka_1 - (\lambda_0 + \lambda_2 + \lambda_4)k - \lambda_0\lambda_2\lambda_4. \quad (5)$$

Moreover, by using (1), we get that

$$\langle (x - \lambda_0)(x - \lambda_2)(x - \lambda_4), 1 \rangle_G = 0 \iff m_1\pi_1 = m_3\pi_3, \quad (6)$$

Thus, from (5) and (6),

$$m_1\pi_1 = m_3\pi_3 \iff (\lambda_1 + 1)(\lambda_3 + 1) = a_1 + 1 - k = -b_1 \quad (7)$$

since  $c_1 = 1$ . Reasoning in the same way with  $\langle (x - \lambda_0)(x - \lambda_1)(x - \lambda_3), 1 \rangle_G = 0$ , we get:

$$m_2\pi_2 = m_4\pi_4 \iff (\lambda_2 + 1)(\lambda_4 + 1) = -b_1.$$

Then, the characterization in (2) follows from Lemma 1(b). To prove the condition in (3), observe that, from the above and Lemma 1(a) it suffices to show that  $m_4\pi_4 = m_0\pi_0$  or, equivalently,  $\langle (x - \lambda_1)(x - \lambda_2)(x - \lambda_3), 1 \rangle_G = 0$ . Now, this leads to the equality

$$k(\lambda_1 + \lambda_2 + \lambda_3) + \lambda_1\lambda_2\lambda_3 = ka_1 = k(k - b_1 - 1)$$

which, together with (7), gives  $k(k + \lambda_1\lambda_3) = \lambda_2(k + \lambda_1\lambda_3)$ . But this can only occur when  $k + \lambda_1\lambda_3 = 0$  or, equivalently,  $\lambda_1 + \lambda_3 = a_1$  (use (7) again). This completes the proof.  $\square$

For the case of bipartite graphs, the conditions in (3) clearly hold since  $\lambda_3 = -\lambda_1$  and  $a_1 = 0$ . Thus, every bipartite strongly distance-regular graph is antipodal. Besides, the condition (2) turns to be very simple:

**Corollary 3.** *A bipartite distance-regular graph  $G$  with diameter  $d = 4$  is strongly distance-regular if and only if  $\lambda_1 = \sqrt{k}$ .*

*Proof.* Apply (2) with  $\lambda_2 = 0$ ,  $\lambda_3 = -\lambda_1$ , and  $\lambda_4 = -k$ .  $\square$

Then, these graphs have spectrum

$$\{k^1, \sqrt{k}^{n/2-k}, 0^{2k-2}, -\sqrt{k}^{n/2-k}, -k^1\}$$

and, in fact, they constitute a well known infinite family (see Brouwer, Cohen and Neumaier [2, p. 425]: With  $n = 2m^2\mu$  and  $k = m\mu$ , they are precisely the incidence graphs of symmetric  $(m, \mu)$ -nets, with intersection array

$$\{k, k-1, k-\mu, 1; 1, \mu, k-1, k\}.$$

We finish this note with a question: Looking at the comprehensive table of distance-regular graphs in [2], it turns out that all the strongly distance-regular graphs (bipartite or not)

with diameter four are antipodal. Thus, at first sight, it seems that conditions (2) and (3) could be closely related (although we have not been able to prove it). Is that the case?

**Acknowledgments.** This note was written while the author was visiting the Department of Combinatorics and Optimization (C&O), in the University of Waterloo (Ontario, Canada). The author sincerely acknowledges to the Department of C&O the hospitality and facilities received. Also, special thanks are due to Chris Godsil, with whom the author discussed the possibility of a characterization liked the one presented.

Research supported by the *Ministerio de Ciencia e Innovación*, Spain, and the *European Regional Development Fund* under project MTM2011-28800-C02-01, and the *Catalan Research Council* under project 2009SGR1387.

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